Exercise 5

Verify the Cauchy–Schwarz inequality and the triangle inequality for the vectors in Exercises 3 to 6.

$$\mathbf{x} = (1, -1, 1, -1, 1),\ \mathbf{y} = (3, 0, 0, 0, 2)$$

Solution

Cauchy–Schwarz Inequality

Check the Cauchy–Schwarz inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ for the given vectors.

$$|\mathbf{x} \cdot \mathbf{y}| = |(1)(3) + (-1)(0) + (1)(0) + (-1)(0) + (1)(2)| = |5| = 5$$

$$\|\mathbf{x}\| = \sqrt{1^2 + (-1)^2 + 1^2 + (-1)^2 + 1^2} = \sqrt{5}$$

$$\|\mathbf{y}\| = \sqrt{3^2 + 0^2 + 0^2 + 0^2 + 2^2} = \sqrt{13}$$

As a result,

$$|\mathbf{x} \cdot \mathbf{y}| = 5 \leq \sqrt{5}\sqrt{13} = \|\mathbf{x}\|\|\mathbf{y}\|,$$

which means the Cauchy–Schwarz inequality is satisfied.

Triangle Inequality

Now check the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for the given vectors.

$$\mathbf{x} + \mathbf{y} = (1, -1, 1, -1, 1) + (3, 0, 0, 0, 2) = (4, -1, 1, -1, 3)$$

$$\|\mathbf{x} + \mathbf{y}\| = \sqrt{4^2 + (-1)^2 + 1^2 + (-1)^2 + 3^2} = \sqrt{28} = 2\sqrt{7}$$

$$\|\mathbf{x}\| = \sqrt{1^2 + (-1)^2 + 1^2 + (-1)^2 + 1^2} = \sqrt{5}$$

$$\|\mathbf{y}\| = \sqrt{3^2 + 0^2 + 0^2 + 0^2 + 2^2} = \sqrt{13}$$

As a result,

$$\|\mathbf{x} + \mathbf{y}\| = 2\sqrt{7} \leq \sqrt{5} + \sqrt{13} = \|\mathbf{x}\| + \|\mathbf{y}\|,$$

which means the triangle inequality is satisfied.